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Institute of Mathematical Sciences
Division of Electromagnetic Research

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On Some Fredholm Integral Equations Arising in Diffraction Theory

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Abstract

Kernels of integral equations arising in diffraction theory are investigated. These kernels depend on a parameter α. Fredholm's theory shows that the resolvents of the kernels are meromorphic functions of α and, hence, can be expanded into power series of α which are convergent for certain regions. Regions of convergence for α, in which the perturbation method would work, are examined.

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In order to show that, in principle, it might be that our Fredholm equations have a unique solution for all α, we give an example of an integral equation of the type studied here for which this is actually the case.
(See 4. Remarks and Examples, p. 11)

2. Integral Equations

Let us consider the following integral equations which were derived by Magnus [5] and Jones [2], respectively.

$$(1) \qquad S(x) + \int_0^1 G(x,y;\alpha)\, S(y)\, dy = T(x)$$

where

$$G(x,y;\alpha) = \frac{-\alpha i}{\pi} \int_0^1 \sqrt{1 - w^2} \left\{ e^{i\alpha w|x-y|} - e^{-i\alpha w(x+y)} \right\} \frac{dw}{w} \ ;$$

$$(2) \qquad S(x) + \int_{-1}^1 K(x,y;\alpha)\, S(y)\, dy = F(x) \ , \quad S(-x) = -\, S(x)$$

where

$$K(x,y;\alpha) = \frac{-i}{\pi} \, \frac{\sinh \alpha(x - y)}{x - y} \ .$$

$S(x)$ is unknown and $T(x)$, $F(x)$ are given functions.

In these integral equations, we find that the kernels contain a parameter α in a non-linear manner. It should also be noted that both kernels are symmetric in x and y but that they are neither hermitian symmetric nor normal, except for special values of α.

It is known from the Fredholm theory that (1) and (2) have the unique

solution $S(x) \in C(\text{or } L^2)$ if $T(x), F(x) \in C(\text{or } L^2)$ and if the homogeneous equations have no solution $S(x) \in C(\text{or } L^2)$ except the trivial solution, where C and L^2 are classes of functions which are continuous and square summable, respectively.

In other words, if α is not an eigenvalue of one of the integral equations, there exists a unique solution for that equation. If this is the case, then the unique solutions are, respectively:

$$(3) \qquad S(x) = T(x) - \int_0^1 R_G(x,y;\alpha)\, T(y)\, dy$$

$$(4) \qquad S(x) = F(x) - \int_{-1}^1 R_K(x,y;\alpha)\, F(y)\, dy$$

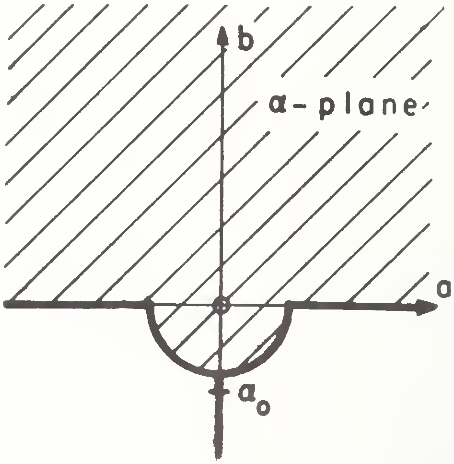
where $R(x,y;\alpha)$ is the resolvent for the respective kernels. The following results are obtained.

<u>The case of Magnus</u> [5]:

There is no eigenvalue

$$\text{for} \quad \begin{cases} \operatorname{Im} \alpha \geq 0 \\ \operatorname{Re} \alpha = 0 \\ |\alpha| \leq \alpha_0 . \end{cases}$$

α_0 can be taken as 1.24, but the actual least upper bound for $|\alpha|$ is unknown.

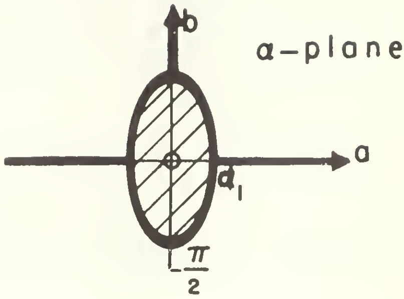


The case of Jones [2]:

There is no eigenvalue

for $\begin{cases} \mathrm{Im}\ \alpha = 0 \\ \mathrm{Re}\ \alpha = 0 \text{ and } b \geq 0 \\ |\alpha| \leq \alpha_1 , \end{cases}$

where $\alpha_1 \geq 1.19$, but the least upper bound for $|\alpha|$ is unknown.



3. Proof of the Results Obtained

The case of Magnus [5]: Let $S = u + iv$, $\frac{1}{\alpha} G = K + iL$, $\alpha = a + ib$ where u, v, K, L, a and b are all real. Let us also define $Hf = \int_0^1 H(x,y)\, f(y)\, dy$, $(f,g) = \int_0^1 f(x)\, \overline{g(x)}\, dx$, where the bar indicates the complex conjugate. Then we have, from (1), by putting $T = 0$,

$$(S,S) + (a + ib)\Big((K + iL)S,S\Big) = 0 .$$

Separating the real and imaginary parts, we obtain

$$(5) \qquad \begin{cases} (S,S) + a(KS,S) - b(LS,S) = 0 \\[2ex] \qquad\qquad b(KS,S) + a(LS,S) = 0 . \end{cases}$$

Solving for (KS,S) and (LS,S), we obtain (for $|\alpha| \neq 0$)

$$(6)\quad \begin{cases} (KS,S) = \dfrac{-a}{|\alpha|^2}\,(S,S) \\[2ex] (LS,S) = \dfrac{b}{|\alpha|^2}\,(S,S)\,. \end{cases}$$

From (1), we also obtain

$$(7)\quad \begin{cases} L = L(x,y;a,b) = \dfrac{1}{\pi}\displaystyle\int_0^1 \sqrt{1-w^2}\left\{e^{-bw(x+y)}\cos aw(x+y) - e^{-bw|x-y|}\cos aw(x-y)\right\}\dfrac{dw}{w} \\[3ex] K = K(x,y;a,b) = \dfrac{-1}{\pi}\displaystyle\int_0^1 \sqrt{1-w^2}\left\{e^{-bw(x+y)}\sin aw(x+y) - e^{-bw|x-y|}\sin aw|x-y|\right\}\dfrac{dw}{w} \end{cases}$$

If we define

$$(8)\quad \begin{cases} T_1(x,\rho) = \begin{cases} e^{bw\rho}\cos awx\,, & \text{if } x > \rho \\ 0\,, & \text{if } x \le \rho \end{cases} \\[4ex] T_2(x,\rho) = \begin{cases} e^{bw\rho}\sin awx\,, & \text{if } x > \rho \\ 0\,, & \text{if } x \le \rho\,. \end{cases} \end{cases}$$

We now obtain

$$(9)\quad L(x,y;a,b) = -\int_0^1 \sqrt{1-w^2}\; e^{-bw(x+y)}\left\{b\int_0^1 \Big(T_1(x,\rho)T_1(y,\rho)+T_2(x,\rho)T_2(y,\rho)\Big)\,d\rho + \frac{1}{w}\sin awx\,\sin awy\right\}dw$$

and

$$(10) \qquad (LS,S) = (Lu,u) + (Lv,v)$$

$$= -\int_0^1 \sqrt{1-w^2} \left\{ b \int_0^1 \left(\Phi_1^2 + \Phi_2^2 + \Psi_1^2 + \Psi_2^2 \right) d\rho + \frac{1}{w} (F_1^2 + F_2^2) \right\} dw$$

where

$$(11) \quad \begin{cases} \Phi_i = \Phi_i(x,\rho,w) = \displaystyle\int_0^1 e^{-bwx} T_i(x,\rho)\, u(x)\, dx & \\ & (i = 1,2) \\ \Psi_i = \Psi_i(x,\rho,w) = \displaystyle\int_0^1 e^{-bwx} T_i(x,\rho)\, v(x)\, dx & \\ F_1 = F_1(x,w) = \displaystyle\int_0^1 e^{-bwx} \sin awx\, u(x)\, dx & \\ F_2 = F_2(x,w) = \displaystyle\int_0^1 e^{-bwx} \sin awx\, v(x)\, dx . & \end{cases}$$

Hence $(LS,S) \leq 0$ if $b \geq 0$; but from (6), $b \geq 0$ implies $(LS,S) \geq 0$ and it follows that $(LS,S) = 0$ if $b \geq 0$. Therefore, $(S,S) = 0$ if $b > 0$, by (6). If $b = 0$, we obtain, from (5), $(LS,S) = 0$ ($a \neq 0$, otherwise trivial) which implies $F_1 = F_2 = 0$ by (10). It follows that $u = v = 0$.

If $a = 0$, we obtain $(S,S) = b(LS,S)$ by (5) and, therefore,

$$(S,S) + b^2 \int_0^1 \sqrt{1-w^2} \int_0^1 (\Phi_1^2 + \Phi_2^2)\, d\rho\, dw = 0$$

by (10) and (11). Since both terms on the left-hand side are non-negative, we obtain $(S,S) = 0$.

It is known that if $\int_0^1\int_0^1 |G(x,y;\alpha)|^2 dxdy < 1$, then the Neumann series of (1) is convergent; that is, for α satisfying the above condition, R_G is regular and α is not an eigenvalue of (1). Since

$$G(x,y;\alpha) = -\frac{\alpha^2}{\pi}\int_0^1 \sqrt{1-w^2}\int_{|x-y|}^{(x+y)} e^{i\alpha w\rho}\, d\rho\ ,$$

$$|G(x,y;\alpha)|^2 = G(x,y;\alpha)\ \overline{G(x,y;\alpha)}$$

$$= \frac{|\alpha|^4}{\pi^2}\int_0^1 \sqrt{1-w^2}\int_{|x-y|}^{(x+y)} e^{i\alpha w\rho}\, d\rho \int_0^1 \sqrt{1-u^2}\int_{|x-y|}^{(x+y)} e^{-i\bar{\alpha}u\lambda}\, d\lambda$$

$$\leq \frac{|\alpha|^4}{\pi^2}\left(\int_0^1 \sqrt{1-w^2}\, dw\int_{|x-y|}^{(x+y)} e^{-bw\rho}\, d\rho\right)^2$$

$$\leq \frac{|\alpha|^4}{(4b)^2}\left(e^{-b(x+y)} - e^{-b|x-y|}\right)^2,\ (b < 0)\ ;$$

therefore,

$$\int_0^1\int_0^1 |G(x,y;\alpha)|^2 dx\, dy \leq \frac{|\alpha|^4}{(4|b|)^2}\int_0^1\int_0^1 \left(e^{|b|(x+y)} - e^{|b||x-y|}\right)^2 dx\, dy$$

$$\leq \frac{4|\alpha|^4}{(4|b|)^4}\left\{e^{4|b|} + 4(1-2|b|)e^{2|b|} - 4|b| - 5\right\}$$

$$= f(a,|b|)\ .$$

By putting $f(a,|b|) < 1$, we find that

$$\begin{cases} \text{when } b = 0 \ , \ a = 2.23 \\ \text{when } a = 0 \ , \ b = 1.24 \ . \end{cases}$$

The case of Jones[2]: Let $S = u + iv$, $K = A + iB$, $\alpha = a + ib$, where u,v,A,B, a and b are all real as before. Then from (2), $F = 0$, $(S,S)+((A+iB)S,S) = 0$; separating the real and imaginary parts, we obtain

$$\begin{cases} (S,S) + (AS,S) = 0 \\ \qquad\quad (BS,S) = 0 \end{cases} \tag{12}$$

where A and B are as follows:

$$\begin{cases} A = A(x,y;a,b) = \frac{1}{\pi} \cosh a(x-y) \frac{\sin b(x-y)}{x-y} = \frac{b}{2\pi} \cosh a(x-y) \int_{-1}^{1} \cos b(x-y)t\,dt \\ B = B(x,y;a,b) = \frac{-1}{\pi} \frac{\sinh a(x-y)}{x-y} \cos b(x-y) \ . \end{cases} \tag{13}$$

(13) is derived from

$$K = A + iB = \frac{-i}{2\pi} \frac{e^{a(x-y)} e^{ib(x-y)} - e^{-a(x-y)} e^{-ib(x-y)}}{x-y} \ .$$

If $b = \operatorname{Im} \alpha = 0$, then from (13), we have $A = 0$. Hence, $(S,S) = 0$ by (12). There is no solution except the trivial solution for $\operatorname{Im} \alpha = 0$. If $a = \operatorname{Re} \alpha = 0$, then

$$A = \frac{b}{2\pi} \int_{-1}^{1} \cos b(x-y)t\,dt \ ,$$

by (13). It follows that

$$(AS,S) = (Au,u) + (Av,v)$$
$$= \frac{b}{2\pi} \int_{-1}^{1} (\Phi_1^{\,2} + \Phi_2^{\,2} + \Psi_1^{\,2} + \Psi_2^{\,2})\, dt$$

where

$$\Phi_1 = \int_{-1}^{1} \cos bxt\; u(x)\, dx \quad , \quad \Phi_2 = \int_{-1}^{1} \sin bxt\; u(x)\, dx$$

$$\Psi_1 = \int_{-1}^{1} \cos bxt\; v(x)\, dx \quad , \quad \Psi_2 = \int_{-1}^{1} \sin bxt\; v(x)\, dx \quad .$$

Hence, $(AS,S) \geq 0$ if $b \geq 0$, in which case we obtain $(S,S) = 0$ from (12).

From

$$|K(x,y)|^2 = \frac{1}{\pi^2} \frac{\sinh \alpha(x-y)}{x-y} \; \frac{\sinh \bar{\alpha}(x-y)}{x-y}$$

$$= \frac{1}{4\pi^2(x-y)^2} \left(e^{\alpha(x-y)} - e^{-\alpha(x-y)}\right)\left(e^{\bar{\alpha}(x-y)} - e^{-\bar{\alpha}(x-y)}\right)$$

$$= \frac{1}{2\pi^2(x-y)^2} \Big(\cosh 2a(x-y) - \cos 2b(x-y)\Big)$$

$$= \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \left\{(2a)^{2n} + (-1)^{n+1}(2b)^{2n}\right\} \frac{(x-y)^{2n-2}}{(2n)!}$$

we obtain:

$$\int_{-1}^{1}\int_{-1}^{1} |K(x,y)|^2 dxdy = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \left\{ (2a)^{2n} + (-1)^{n+1}(2b)^{2n} \right\} \frac{1}{(2n)!}$$

$$\times \int_{-1}^{1}\int_{-1}^{1} (x-y)^{2n-2}\, dxdy$$

$$= \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \left\{ (2a)^{2n} + (-1)^{n+1}(2b)^{2n} \right\} \frac{1}{(2n)!} \frac{2^{2n+1}}{(2n-1)2n}$$

$$\leq f(a,b)$$

where

$$f(a,b) = \frac{1}{\pi^2} \left[\frac{1}{12} (\cosh 4a + \cos 4b) + \frac{10}{3} (a^2 - b^2) + \frac{1}{15} \right.$$

$$\left. \times \left(\cosh 4b - 1 - \frac{(4b)^2}{2!} - \frac{(4b)^4}{4!} \right) \right] .$$

Putting $f(a,b) < 1$, we obtain $a \leq 1.19$ when $b = 0$ and $b \leq 1.5$ when $a = 0$.

If we take

$$f(a,b) = \frac{1}{\pi^2} \left(\frac{1}{2} \cosh 4a - \frac{1}{2} + 4b^2 \right) ,$$

then we have $b < \frac{\pi}{2}$ when $a = 0$.

4. Remarks and Examples

We may summarize the methods used here as follows: Let us consider the following integral equation

$$S(x) + \int_0^1 G(x,y;\alpha)\, S(y)\, dy = T(x) \tag{14}$$

with the kernel $G(x,y;\alpha) = 2\alpha\, e^{\alpha(x+y)}$.

We can easily find the Fredholm determinant, its First Minor and the resolvent of (14) as follows:

$$D(\alpha) = e^{2\alpha}$$

$$D(x,y;\alpha) = 2\alpha\, e^{\alpha(x+y)}$$

$$R(x,y;\alpha) = \frac{D(x,y;\alpha)}{D(\alpha)} = 2\alpha\, e^{\alpha(x+y-2)} .$$

In this case, the integral equation does not have any eigenvalue for any finite α since the Fredholm determinant $D(\alpha)$ is an entire function which has no zeros for any finite α, and we have the unique solution:

$$S(x) = T(X) - \int_0^1 2\alpha\, e^{\alpha(x+y-2)}\, T(y)\, dy .$$

In the example above, it is easy to compute the Fredholm determinant: however, we are dealing here with a rather special case. In most cases (see Magnus and Jones), it is very difficult to evaluate the Fredholm determinant. Since we know that zeros of the Fredholm determinant are eigenvalues of the integral equation from which it arises, one method for locating eigenvalues is to investigate zeros of the Fredholm determinant.

If an evaluation of Fredholm's determinant is difficult, an indirect method proving non-existence of eigenvalues for certain regions is useful. If the real or imaginary part of a kernel considered is positive (or negative) definite, the indirect method would work out in general (see Magnus' case for $\mathrm{Im}\,\alpha \geq 0$, $\mathrm{Re}\,\alpha = 0$; Jones' case for $\mathrm{Im}\,\alpha = 0$ and $\mathrm{Re}\,\alpha = 0$, $b \geq 0$).

Another indirect method is to obtain a majorant $f(a,b)$ of $\iint |K(x,y;a,b)|^2 dxdy$ (see Section 3) and to find the region $R = \left\{ (a,b) \,|\, f(a,b) < 1 \right\}$ of non-existence of eigenvalues. The region obtained by this method is, however, limited to the neighborhood of the origin and varies according to the majorant function used.

It should also be noted that for some kernels of the type considered (symmetric but non-linear with respect to the parameter), there exist no eiganvalues even if we have a real symmetric kernel. This cannot happen if $K(x,y;\alpha)=\alpha K(x,y)$, where $K(x,y)$ is independent of α since we know that there then exists at least one real eigenvalue α (e.g. See [6]).

We shall now consider another example arising from the work of Noble[7]:

$$W(x) + \int_0^1 y\, W(y)\, K_\mu(x,y;k)\, dy = F(x) \quad (0 < x < 1) \tag{15}$$

where

$$K_\mu(x,y;k) = \begin{cases} -i \displaystyle\int_0^k H_\mu^{(1)}(xt) J_\mu(yt) \sqrt{k^2 - t^2}\, dt, & x > y \\ \\ -i \displaystyle\int_0^k H_\mu^{(1)}(yt) J_\mu(xt) \sqrt{k^2 - t^2}\, dt, & y > x \end{cases}$$

the integral equation (15) is one of Fredholm's type, and symmetric in x and y with the weight function y.

Let $W = u + iv$, $K_\mu = S_\mu + iT_\mu$ (for k real), then we obtain from (15), by putting $F = 0$:

$$(16)\quad \begin{cases} u(x) + \displaystyle\int_0^1 y\left\{S_\mu(x,y;k)u(y) - T_\mu(x,y;k)v(y)\right\} dy = 0 \\[2ex] v(x) + \displaystyle\int_0^1 y\left\{S_\mu(x,y;k)v(y) + T_\mu(x,y;k)u(y)\right\} dy = 0 \end{cases}$$

where

$$T_\mu(x,y;k) = -\int_0^k \sqrt{k^2 - t^2}\; J_\mu(xt)\, J_\mu(yt)\, dt$$

since

$$H_\mu^{(1)}(xt) = J_\mu(xt) + iN_\mu(xt).$$

After eliminating S_μ from (16), we obtain

$$(17)\quad \int_0^1\int_0^1 T_\mu(x,y;k)\left\{yu(y)\, xu(x) + yv(y)\, xv(x)\right\} dy\, dx = 0\ .$$

Changing the order of integration, we obtain

$$\int_0^k \sqrt{k^2 - t^2}\left\{\left(\int_0^1 xu(x)J_\mu(xt)dx\right)^2 + \left(\int_0^1 xv(x)J_\mu(xt)dx\right)^2\right\} dt = 0.$$

Hence

$$(18)\quad \begin{cases} \displaystyle\int_0^1 xu(x)J_\mu(xt)dx = 0 \\[2ex] \displaystyle\int_0^1 xv(x)J_\mu(xt)dx = 0\ , \end{cases}$$

and it follows that

$$(19)\qquad \begin{cases} \displaystyle\int_0^1 x^{1+\mu+2n}\, u(x)dx = 0 \\ \displaystyle\int_0 x^{1+\mu+2n}\, v(x)dx = 0 \ . \end{cases} \qquad (n = 0,1,2,\ldots)$$

We obtain $u = v = 0$, by (19), whenever μ is real and $\mu > -2$.

When $\mu = \frac{1}{2}$, (15) can be reduced to Magnus' equation (1) by setting $S(x) = \sqrt{x}\, W(x)$. In this case,

$$K_{\frac{1}{2}}(x,y;k) = \frac{-i}{\pi} \frac{1}{\sqrt{xy}} \int_0^k \frac{\sqrt{k^2-t^2}}{t} \left\{ e^{i|x-y|t} - e^{i(x+y)t} \right\} dt$$

$$= \frac{-ik}{\pi} \frac{1}{\sqrt{xy}} \int_0^1 \frac{\sqrt{1-t^2}}{t} \left\{ e^{ikt|x-y|} - e^{ikt(x+y)} \right\} dt$$

$$= G(x,y;k)/\sqrt{xy} \ .$$

Appendix I

Let us consider the following integral equation:

$$(20) \qquad S(x) + \lambda \int_a^b G(x,y;\alpha)S(y)dy = T(x) .$$

By Fredholm's theory, we obtain the determinant $D_\alpha(\lambda)$, the First Minor $D_\alpha(x,y;\lambda)$ and the resolvent $R_\alpha(x,y;\lambda)$ as follows:

$$(21) \quad D_\alpha(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n\,!} \underbrace{\int_a^b \cdots \int_a^b}_{n} \begin{vmatrix} G(t_1,t_1;\alpha), \ldots, G(t_1,t_n;\alpha) \\ \ldots\ldots\ldots, \ldots, \ldots\ldots\ldots \\ \ldots\ldots\ldots, \ldots, \ldots\ldots\ldots \\ G(t_n,t_1;\alpha), \ldots, G(t_n,t_n;\alpha) \end{vmatrix} dt_1 \ldots dt_n$$

$$(22) \quad D_\alpha(x,y;\lambda) = G(x,y;\alpha)$$

$$+ \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n\,!} \underbrace{\int_a^b \cdots \int_a^b}_{n} \begin{vmatrix} G(x,y;\alpha), G(x,t_1;\alpha), \ldots, G(x,t_n;\alpha) \\ \ldots\ldots\ldots, \ldots\ldots\ldots, \ldots, \ldots\ldots\ldots \\ \ldots\ldots\ldots, \ldots\ldots\ldots, \ldots, \ldots\ldots\ldots \\ G(t_n,y;\alpha), G(t_n,t_1;\alpha), \ldots, G(t_n,t_n;\alpha) \end{vmatrix} dt_1 \ldots dt_n$$

$$(23) \quad R_\alpha(x,y;\lambda) = D_\alpha(x,y;\lambda)/D_\alpha(\lambda) .$$

If $\lambda = -1$, then (20) is of the type considered here. Therefore, we define $D(\alpha)$, $D(x,y;\alpha)$ and $R(x,y;\alpha)$ by the identities:

$$(24) \qquad D(\alpha) \equiv D_\alpha(-1)$$

$$(25) \qquad D(x,y;\alpha) \equiv D_\alpha(x,y;-1)$$

$$(26) \qquad R(x,y;\alpha) \equiv R_\alpha(x,y;-1) .$$

If $G(x,y;\alpha)$ is an entire function of α, then $D(\alpha)$, $D(x,y;\alpha)$ are entire functions of α and $R(x,y;\alpha)$ is a meromorphic function of α.

Since $D(\alpha)$ is an entire function, Picard's theorem* shows that $D(\alpha)$ has infinitely many zeros unless zero is the exceptional value. Therefore, except in this case, $R(x,y;\alpha)$ has infinitely many poles which are eigenvalues of the equation (20) with $\lambda = -1$.

Since $D(\alpha)$, $D(x,y;\alpha)$ are entire functions, $D(x,y;\alpha)$ can be expanded into a power series in α which converges uniformly and absolutely for all finite α; $1/D(\alpha)$ can be expanded into a power series in α which converges uniformly and absolutely for $|\alpha| < \min_i |\alpha_i|$ where α_i are the zeros of $D(\alpha)$ (i.e., eigenvalues).

Hence, $R(x,y;\alpha)$ can be expanded into a power series in α which is uniformly and absolutely convergent for $|\alpha| < \min_i |\alpha_i|$.

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* See, e.g., G. Valiron - Theory of Integral Functions; New York, Chelsea Publ. Co.

Appendix II

For the case of diffraction by a circular aperture, the eigenvalues and eigenfunctions have been calculated approximately by means of the geometrical theory of diffraction. This method shows that the eigenvalues are asymptotically equal to the roots of the following equations (See [3], pp. 435-438)

$$(2\pi\alpha)^{\frac{1}{2}} e^{-i(2\alpha + \pi/4)} + 1 = 0. \tag{27}$$

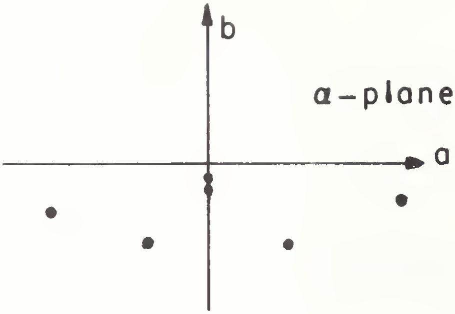
Equation (27) has infinitely many roots which are

$$\alpha = -\ .14i\ ,\ -\ .41i$$

$$\alpha_n = \frac{\gamma_n}{4} e^{i(\frac{3\pi}{2} \pm \vartheta_n)} \qquad (\vartheta_n \to \frac{\pi}{2} \text{ as } n \to \infty)$$

where

$$\begin{cases} \vartheta_n + 2n\pi = \gamma_n \sin \vartheta_n \\ \log \pi\, \gamma_n = \gamma_n \cos \vartheta_n . \end{cases}$$



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